UDC 62-50

# CONTROL SYNTHESIS IN A PROBLEM WITH PHASE CONSTRAINTS

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We consider the time-optimal control of a plant which is described by a system of linear fourth-order differential equations with constant coefficients, on which certain constraints are imposed; there are two phase constraints. We describe the properties of the optimal control and realize its synthesis.

**1. Statement of the problem** (*). The problem of controlling a plant described by the differential equations

$$d\delta/dt = u, \qquad dF/dt = a_2F + au + a_1\delta$$
$$d\vartheta/dt = \vartheta_1, \qquad d\vartheta_1/dt = b_4\vartheta_1 + b_2F \tag{1.1}$$

with constant coefficients and with the constraints

$$u^2 \leqslant u_0^2, \quad \delta^2 \leqslant \delta_0^2, \qquad F^2 \leqslant F_0^2 \tag{1.2}$$

is a characteristic one in a number of applications. The coefficients of system (1.1) are related by

$$a = \frac{k_fT_1}{T_2}, \quad a_1 = \frac{k_f}{T_2}, \quad a_2 = -\frac{1}{T_2}, \quad b_2 = \frac{k_0}{T_2}, \quad b_4 = -\frac{1}{T_1} \tag{1.3}$$

The quantities $u_0$, $\delta_0$, $F_0$ are constant and positive. It is assumed that

$$a_2 < 0, \quad b_4 < 0, \quad T_1 > T_2 > 0, \quad k_0 > 0, \quad k_f > 0$$

We need to find a measurable function $u^\circ(t)$ (the optimal control) which with maximal rapidity takes the plant from the initial state $\delta = F = \vartheta = \vartheta_1 = 0$ to the terminal state $\delta = F = \vartheta_1 = 0$, $\vartheta = \vartheta_k$ and ensures the fulfillment of constraints (1.2).

**2. Optimality conditions.** We investigate this problem by using the results in [1]. Let $\delta^\circ(t)$, $F^\circ(t)$, $\vartheta^\circ(t)$, $\vartheta_1^\circ(t)$, $u^\circ(t)$ be the optimal solution. We write out the maximum principle,

$$u^\circ(t) = u_0 \operatorname{sign}(\psi_1 + a\psi_2) \tag{2.1}$$

$$-\frac{d\psi_1}{dt} = a_1\psi_2 - 2\delta^\circ\frac{d\mu_1}{dt}, \qquad -\frac{d\psi_2}{dt} = a_2\psi_2 + b_2\psi_4 - 2F^\circ\frac{d\mu_2}{dt}$$
$$-\frac{d\psi_3}{dt} = 0, \qquad -\frac{d\psi_4}{dt} = \psi_3 + b_4\psi_4 \tag{2.2}$$

$$\frac{d\mu_1}{dt}[(\delta^\circ(t))^2 - \delta_0^2] = 0, \qquad \frac{d\mu_1}{dt} \geqslant 0$$
$$\frac{d\mu_2}{dt}[(F^\circ(t))^2 - F_0^2] = 0, \qquad \frac{d\mu_2}{dt} \geqslant 0 \tag{2.3}$$

We obtain an expression for the function $\psi_1 + a\psi_2$ from the solution of differential equations (2.2),

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*) The problem was posed by N. P. Dergunov.

$$\psi_1(t) + a\psi_2(t) = (\psi_1 + a\psi_2)_\mu + Q(t)$$
$$(\psi_1 + a\psi_2)_\mu = A_1 e^{-b_4 t} + A_2 e^{-a_2 t} + A_3 t + A_4$$
$$Q(t) = \int_0^t \left[2\delta^\circ(t)\frac{d\mu_1}{dt} + 2aF^\circ(t)\frac{d\mu_2}{dt}\right]dt - (a_1 + aa_2)\times$$
$$\int_0^t e^{-a_2\tau}\int_0^\tau 2F^\circ(\eta)\frac{d\mu_2}{d\eta}e^{a_2\eta}\,d\eta \tag{2.4}$$
$$A_1 = \frac{(a_1 + ab_4)\,b_2}{a_2 - b_4}C_4, \qquad A_2 = \frac{a_1 + aa_2}{a_2}C_2$$
$$A_3 = -\frac{a_1 b_2}{a_2 b_4}C_3, \qquad A_4 = C_4$$

where $C_1$, $C_2$, $C_3$, $C_4$ are arbitrary constants. Using the relations for the coefficients of the system of differential equations (1.1), we obtain $A_1 = 0$.

**3. Structure of the optimal solutions.** 1°. If the optimal solution is such that the measures $d\mu_i/dt = 0$ $(i = 1, 2)$, then it coincides with the time-optimal solution of the corresponding problem without phase constraints. From (2.4) it follows that in this case the optimal control has two switchings and is a piecewise-constant time function. On the ( $y_1 y_2$ )-plane, where

$$y_1 = \frac{a_2 - a_1}{a_2}\delta, \qquad y_2 = F + \frac{a_1}{a_2}\delta$$

the phase trajectories of the system with $u = \pm u_0$ are determined by the expression

$$y_2 \pm \frac{aa_2 + a_1}{a_2^2}u_0 = C\exp\left[\frac{a_2^2}{\pm(a_2 - a_1)\,u_0}y_1\right] \tag{3.1}$$

Here $C$ is an arbitrary constant. In Fig. 1 the dotted curves represent both types of phase trajectories. The limiting curves $\Pi_1$ and $\Pi_2$ (Fig. 1) are determined by the dependencies (3.1) as $y_1 \to \pm\infty$ for $u = \pm u_0$, respectively. The closed curve corresponds to a typical optimal trajectory.

2°. Let the optimal solution be such that at least one of the measures $d\mu_1/dt$, $d\mu_2/dt$ is not identically equal to zero. Let us study function (2.4). By a lemma from [2] the function $(\psi_1 + a\psi_2)_\mu$ has not more than two zeros,

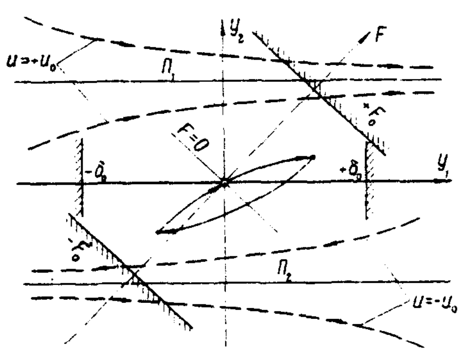


Fig. 1

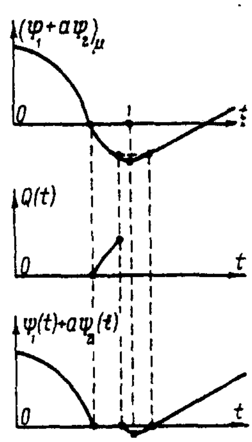


Fig. 2

while its derivative has not more than one zero. The forms of this function, of the function $Q(t)$, and of $\psi_1(t) + a\psi_2(t)$ are shown in Fig. 2. We note that in Fig. 2 we have indicated and examined the case when the function $\psi_1(t) + a\psi_2(t)$ is positive for $t = 0$. The discussion is analogous for the other case.

We return to Fig. 1. In the coordinates $y_1, y_2$ the phase constraints $F = \pm F_0$ map into the straight lines $\pm F_0$ defined by the expression

$$y_1 = (y_2 \mp F_0)\frac{a_2 - a_1}{a_1}$$

while the phase constraints $\delta = \pm\delta_0$ map into the corresponding straight lines $\pm\delta_0$. The system can go onto the phase constraints $+F_0, +\delta_0, (-F_0, -\delta_0)$ (see Fig. 1 wherein the phase constraints are hatched) only with a positive (negative) control. Since $a > 0$ and $a_1 + aa_2 < 0$, we see from expression (2.4) that neither the measure $d\mu_1 / dt$ nor the measure $d\mu_2 / dt$ can have jumps at the instant of going onto a phase constraint, i.e. at the instant of going onto $\psi_1(t) + a\psi_2(t) = 0$. Note also that when it is located on the phase boundary $\psi_1(t) + a\psi_2(t) = 0$ the measure cannot have jumps at the instant of leaving from the phase constraint. From all of this it follows (this is shown in Fig. 2 for one case) that the departure from the constraint $+F_0$ or $+\delta_0$ can be effected only at the instants when $d(\psi_1 + a\psi_2)_\mu / dt < 0$, because otherwise the phase constraint is violated (departure from $-F_0$, $-\delta_0$, when $d(\psi_1 + a\psi_2)_\mu / dt > 0$). Thus, when $(\psi_1 + a\psi_2) = (\psi_1 + a\psi_2)_\mu > 0$ holds at the initial instant $t = 0$ the optimal trajectory can go onto the phase constraints $+F_0$, $+\delta_0$ only once, and onto the constraints $-F_0$, $-\delta_0$ only once.

Note 1. Each solution of the differential equations from (1.1) satisfies the relation

$$T_1\frac{d}{dt}(T_2\vartheta_1^{\cdot} + \vartheta_1 - k_0k_f\delta) + (T_2\vartheta_1^{\cdot} + \vartheta_1 - k_0k_f\delta) = 0$$

Hence, taking into account that $\vartheta_1 = F = \delta = 0$ at the terminal point, we obtain

$$(T_2 - T_1)\vartheta_1 = T_2k_0F - T_1k_0k_f\delta \tag{3.2}$$

Note 2. Let us consider the portions of the optimal trajectory, lying on the phase constraints. For $|F^\circ(t)| = F_0$, for example, for $F^\circ(t) \equiv F_0$, from (1.1) we find that $u^\circ(t)$ is a continuously differentiable function

$$u^\circ(t) = -\frac{a_1\delta(t_1) + a_2F_0}{a} e^{-a_1(t-t_1)a^{-1}} \tag{3.3}$$

where $t_1$ is the instant of going onto the phase constraint. Here the coordinate

$$\delta^\circ(t) = \left[\delta(t_1) + \frac{a_2}{a_1}F_0\right]e^{-a_1(t-t_1)a^{-1}} - \frac{a_2}{a_1}F_0 \tag{3.4}$$

and, increasing monotonically, tends to $\delta(\infty) = -a_2F_0 / a_1$. This follows from $a_1/a > 0$, (3.1), and $a_1\delta + a_2F = k_f(T_2 - T_1)T_2^{-1}[1 - e^{-\delta/T_1u_0}] < 0$. For $|\delta^\circ(t)| = \delta_0$, for example, $\delta^\circ = +\delta_0$, $u^\circ(t) \equiv 0$. The coordinate $F^\circ(t)$, decreasing monotonically, tends to $F(\infty) = -a_1\delta_0a_2^{-1} > 0$. Thus, the optimal trajectory goes onto the phase constraints only in the following order: $+F_0$, $+\delta_0$, $-F_0$, $-\delta_0$, although here it may not go onto some of them.

**4. Synthesis of the optimal control.** For forming the control synthesis we restrict ourselves to the case, most often encountered in practice, when the phase trajectory with $u = \pm u_0$, passing through the origin, intersects at first the phase constraint $F = \pm F_0$. The other case possible, when at first the constraint $\delta = \pm\delta_0$ is intersected, is analyzed analogously.

Let us investigate the optimal solution with the control $u^0(t) = \pm u_0$ on the initial segment. From a proposition following later we conclude that if the initial segment of the optimal trajectory terminates within the phase constraints, then the whole trajectory lies within the phase constraints. In this case the synthesis problem is the determination of the two switching instants with the aid of the current phase coordinates of the plant. Let us consider the case when the initial segment of the trajectory terminates on the phase constraint $F = \pm F_0$. Here the synthesis problem consists of determining the instant $t_3$ of leaving from the phase constraint (from the constraint $F = \pm F_0$ or from $\delta = \pm\delta_0$ following it) and, after this, of distinguishing the two possible terminal segments of the optimal trajectory:

1) the terminal segment is wholly located within the phase constraints (Fig. 3; the trajectory passing through the points 2, 1, 0) is of type (1);

2) the terminal segment goes onto the phase constraint $F^0(t) = \mp F_0$ (Fig. 3; the trajectory passing through the points 3, $\delta_2^-$, $\delta_1^-$, 0) is of type (2).

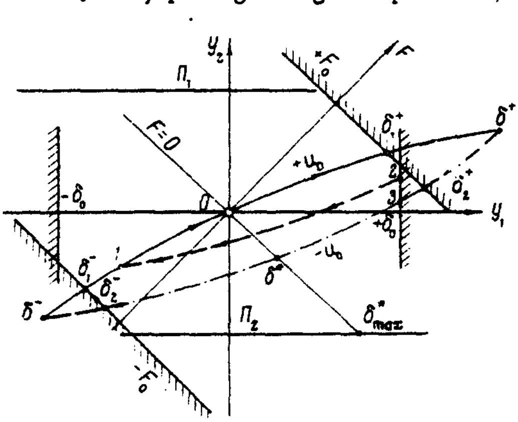


Fig. 3

In case (1) the control synthesis is effected if the instant $t_3$ and the instant $t_5$ – the instant of crossing over to the trajectory segment with $u = \pm u_0$, passing through the origin (Fig. 3) – have been expressed in terms of the system's phase coordinates. The variation of the plant's phase state on the terminal segment is described, with due regard to (3.2), by the following equations:

$$\frac{d\delta}{dt} = u, \qquad \frac{d\vartheta_1}{dt} = a_2\vartheta_1 + a_3\delta$$

$$\frac{d\vartheta}{dt} = \vartheta_1, \qquad a_3 = \frac{k_0 k_f}{T_2} \qquad (4.1)$$

For $u = u_0$ the phase trajectories of this system in the coordinates $\delta^*$, $\vartheta^*_1$, where

$$\delta^* = -\frac{a_3}{a_2}\delta, \qquad \vartheta^*_1 = -\frac{a_3}{a_2}\delta - \vartheta_1 \qquad (4.2)$$

are described by the equation

$$\vartheta_1^* - \frac{a_3}{a_2^2} u_0 = Ce^{-\frac{a_2^2}{a_3 u_0}\delta^*} \qquad (4.3)$$

where $C$ is a parameter.

Let us express the phase coordinates of the system at the instant $t_5$ (we assign them the index (5)) in terms of the coordinates at the instant $t_3$ (the index (3)) of crossing over onto the terminal segment of the optimal trajectory. Using Eq. (4.3) we obtain

$$\vartheta_1^{*(5)} = \mp\left[\mp\frac{a_3}{a_2^2}u_0\left(\vartheta_1^{*(3)} \pm \frac{a_3}{a_2^2}u_0\right)\exp\left(\mp\frac{a_2^2}{a_3u_0}\delta^{*(3)}\right) + \left(\frac{a_3}{a_2}u_0\right)^2\right]^{1/2} \qquad (4.4)$$

$$\delta^{*(5)} = \mp\frac{a_3u_0}{a_2^2}\ln\left|\mp\frac{a_2^2}{a_3u_0}\vartheta_1^{*(5)} + 1\right| \qquad (4.5)$$

Here and below the superscripts correspond to trajectories starting off with $u = +u_0$, while the subscripts – to trajectories strating off with $u = -u_0$. Having integrated Eqs. (4.1) from the terminal point $\delta = F = \vartheta_1 = 0$, $\vartheta = \vartheta_k$ in reverse order, respectively, to the point $t_5$ with $u = \pm u_0$, and from the point $t_5$ to $t_3$ with $u = \mp u_0$, we obtain

$$\vartheta^{(3)} = -\frac{1}{a_2}\left[\vartheta_1^{(5)} + \frac{a_3\delta^{(5)}}{a_2} \mp \frac{a_3u_0}{a_2^2}\right][e^{-a_2\tau^{(3)}} - 1] + \left[-\frac{a_3\delta^{(5)}}{a_2}\tau^{(3)} \mp a_3u_0\left(\frac{(\tau^{(3)})^2}{2a_2} - \frac{\tau^{(3)}}{a_2^2}\right)\right] + \vartheta^{(5)} \tag{4.6}$$

$$\tau^{(3)} = \frac{\delta^{(3)} - \delta^{(5)}}{\pm u_0}, \qquad \tau^{(5)} = \frac{\delta^{(5)}}{\mp u_0}$$

$$\vartheta_1^{(5)} = \pm\frac{a_3u_0}{a_2^2}e^{-a_2\tau^{(5)}} \pm a_3u_0\left[\frac{\tau^{(5)}}{a_2} - \frac{1}{a_2^2}\right]$$

$$\vartheta^{(5)} = \mp\frac{a_3u_0}{a_2^3}[e^{-a_2\tau^{(5)}} - 1] \pm a_3u_0\left[\frac{\tau^{(5)}}{2a_2} - \frac{1}{a_2^2}\right]\tau^{(5)} + \vartheta_k$$

Substituting (4.4), (4.5) and (4.2) into (4.6), we obtain

$$\vartheta^3 = \vartheta_\pm^{(3)}(\delta^{(3)}, \vartheta_1^{(3)}, \vartheta_k)$$

Hence it follows that the instant $t_3$ is determined from the condition that the function

$$f_1(\delta, \vartheta_1, \vartheta, \vartheta_k) = \vartheta(t) - \vartheta_\pm^{(3)}(\delta(t), \vartheta_1(t), \vartheta_k) \tag{4.7}$$

equals zero.

In case (2), besides the instants $t_3$, $t_5$, there is one more characteristic instant $t_4$ namely, the instant of going onto the phase constraint $F = \mp F_0$, which must be expressed in terms of the current phase coordinates. For motion under this constraint the family of phase trajectories of the system is described by the functions:

in the ($\vartheta\vartheta_1$)-plane

$$\vartheta = \pm\frac{F_0}{b_4}\ln\left|\vartheta_1 \mp \frac{b_2}{b_4}F_0\right| + \left(\vartheta_1 \mp \frac{b_2}{b_4}F_0\right)\frac{1}{b_2} + C \tag{4.8}$$

in the ($\delta\vartheta_1$)-plane

$$\delta \pm \frac{a_2}{a_1}F_0 = \frac{C}{\vartheta_1 \mp (b_2/b_4)F_0} \tag{4.9}$$

Moving from the terminal point to point $t_5$, we obtain the functions $\vartheta_1^{(5)}(\delta^{(5)})$, $\vartheta^{(5)}(\delta^{(5)})$. Using the first one of them, we determine $\delta^{(5)}$ for $F = \mp F_0$ with the aid of (3.2),

$$\mp T_2k_0F - T_1k_0k_f\delta^{(5)} - (T_2 - T_1)\vartheta_1^{(5)}(\delta^{(5)}) = 0$$

From (4.8), (4.9) we find that at the point $t_4$

$$\vartheta^{(4)} = \pm\frac{F_0}{b_4}\ln\left|\vartheta_1^{(4)} \mp \frac{b_2}{b_4}F_0\right| + \left(\vartheta_1^{(4)} \mp \frac{b_2}{b_4}F_0\right)\frac{1}{b_2} + C_\vartheta^{(5)} \tag{4.10}$$

$$\delta^{(4)} \pm \frac{a_2}{a_1}F_0 = C_{\vartheta_1}^{(5)} \Big/ \left(\vartheta_1^{(4)} + \frac{b_2}{b_4}F_0\right) \tag{4.11}$$

where the constants $C_\vartheta^{(5)}(\vartheta_k)$, $C_{\vartheta_1}^{(5)}(\vartheta_k)$ are found from the condition that these phase trajectories pass through the point $\delta^{(5)}$, $\vartheta_1^{(5)}$, $\vartheta^{(5)}$, $\mp F_0$.

For motion in the open region, when $u = \text{const} = \mp u_0$, from system (1.1) we can obtain

$$\frac{d}{d\delta}\left(\vartheta_1 - b_4\vartheta - \frac{b_2}{a_2}F + a\frac{b_2}{a_2}\delta\right) = -\frac{a_1 b_2}{a_2 u_0}\delta$$

which corresponds to

$$\mp\frac{a_2 u_0(\vartheta_1 - b_4\vartheta)}{a_1 b_2} \pm \frac{u_0}{a_1}F \mp \frac{a u_0}{a_1}\delta = C^{(4)} - \frac{\delta^2}{2} \tag{4.12}$$

From the condition that the phase trajectory (4.12) passes through the point $\delta^{(4)}$, $\vartheta_1^{(4)}$, $\vartheta^{(4)}$, $\mp F_0$ we find

$$C^{(4)} = \mp\frac{a_2 u_0}{a_1 b_2}(\vartheta_1^{(4)} - b_4\vartheta^{(4)}) \mp \frac{u_0}{a_1}F_0 \mp \frac{a u_0}{a_1}\delta^{(4)} + \frac{(\delta^{(4)})^2}{2} \tag{4.13}$$

With the aid of dependency (3.1) we obtain the equation determining $\delta^{(4)}$

$$\mp F_0 + \frac{a_1}{a_2}\delta^{(4)} \mp \frac{a a_2 + a_1}{a_2^2}u_0 = C^{(3)}\exp\left(\mp\frac{a_2}{u_0}\delta^{(4)}\right) \tag{4.14}$$

$$C^{(3)} = \left[F^{(3)} + \frac{a_1}{a_2}\delta^{(3)} \mp \frac{a a_2 + a_1}{a_2^2}u_0\right]\exp\left(\pm\frac{a_2}{u_0}\delta^{(3)}\right)$$

$$T_2 k_0 F^{(3)} - T_1 k_0 k_f \delta^{(3)} - (T_2 - T_1)\dot{\vartheta}_1^{(3)} = 0$$

Thus, from (4.14), (4.10), (4.11) and (4.13) we obtain, respectively,

$$\delta^{(4)} = \delta^{(4)}\ (\delta^{(3)}, \vartheta_1^{(3)}), \qquad \vartheta^{(4)} = \vartheta^{(4)}\ (\delta^{(3)}, \vartheta_1^{(3)}, \vartheta_k) \tag{4.15}$$

$$\vartheta_1^{(4)} = \vartheta_1^{(4)}(\delta^{(3)}, \vartheta_1^{(3)}), \qquad C^{(4)} = C^{(4)}\ (\delta^{(3)}, \vartheta_1^{(3)}, \vartheta_k)$$

The crossing of the optimal phase trajectory over to the terminal segment, i. e. the determination of the instant $t_3$ in the optimal control synthesis, takes place when the current system phase coordinates $\delta$, $\vartheta_1$, $\vartheta$ satisfy the equation

$$f_2(\delta, \vartheta_1, \vartheta, \vartheta_k) = \mp\frac{a_2 u_0}{a_1 b_2}(\vartheta_1 - b_4\vartheta) \pm \frac{u_0}{a_1}F \mp$$
$$\mp\frac{a u_0}{a_1}\delta - C^{(4)}(\delta^{(3)}, \vartheta_1^{(3)}, \vartheta_k) + \frac{\delta^2}{2} = 0 \tag{4.16}$$

where $C^{(4)}$ (...) is determined from (4.14) and $F$ is determined in terms of $\vartheta_1$, $\delta$ from (3.2).

The control synthesis is constructed in the following manner. During the plant's motion we compute the above-defined (see (4.7), (4.16)) functions of the current phase coordinates and of the required terminal $\vartheta_k$: $f_1$ (...), $f_2$ (...), and we verify the conditions $F(t) = +F_0\ (-F_0)$ and $\delta(t) = +\delta_0\ (-\delta_0)$. At first the plant's motion takes place along the initial trajectory (Fig. 3) with $u^\circ(t) = \pm u_0$. The choice of this or the other starting point is determined by the rule $u^\circ(t)\ \vartheta_k > 0$, which is proved in the Appendix. At the instant that $F(t) = +F_0(-F_0)$ the trajectory goes onto the phase constraint on $F$ and the motion continues with $u^\circ(t)$ determined in accordance with formula (3.3). Here the coordinate $|\delta(t)|$ increases monotonically. At the instant that $\delta(t) = +\delta_0\ (-\delta_0)$ the trajectory goes onto the phase constraint on $\delta$ and the motion continues with $u^\circ(t) \equiv 0$. Right away the trajectory comes off the phase constraint on $F$ since the coordinate $|F|$ decreases monotonically. The object realizes the motion described either completely or partially depending on when the instant $t_3$ of coming off the phase constraints occurs. This instant coincides with the instant of fulfillment of one of the equalities

$$f_1(\delta, \vartheta_1, \vartheta, \vartheta_k) = 0, \qquad f_2(\delta, \vartheta_1, \vartheta, \vartheta_k) = 0$$

In case the first equality is fulfilled the optimal trajectory has a terminal segment of type 1. The plant's motion from the instant $t_3$ takes place with $u^\circ(t) = \mp u_0$ up to the instant $t_5$ and $f_1(\delta(t), \vartheta_1(t), \vartheta(t), \vartheta_k) \equiv 0$. The instant $t_5$ is determined as the first instant after $t_3$ that the function $f_1(\ldots)$ once again becomes nonzero. Beginning with $t_5$, $u^\circ(t) = \pm u_0$ and remains so until the end of the process, determined by the equality $\vartheta(t) = \vartheta_k$.

In case the second equality is fulfilled the optimal trajectory has a terminal segment of type 2. The plant's motion from the instant $t_3$ continues with $u^\circ(t) = \mp u_0$ up to the instant $t_4$ and $f_2(\delta(t), \vartheta_1(t), \vartheta(t), \vartheta_k) \equiv 0$. The instant $t_4$ is determined from the condition $F(t) = -F_0(+F_0)$. Starting from the instant $t_4$ and up to instant $t_5$ the motion takes place with $u^\circ(t)$ determined in accordance with formula (3.3). The instant $t_5$ is determined by the condition $f_1(\delta(t), \vartheta_1(t), \vartheta(t), \vartheta_k) = 0$. We note that up to this instant this function has not changed sign from the beginning of the motion. From instant $t_5$ and up to the end of the process the motion takes place with $u^\circ(t) = \pm u_0$.

**5. Appendix.** Let us examine the differential equation

$$\frac{d}{dt}(\vartheta_1 - b_4\vartheta) = b_2F$$

obtained from (1.1). The function

$$J(t) = \vartheta_1(t) - b_4\vartheta(t) = \int_0^t b_2F(t)\,dt$$

has $J(0) = 0$, $J(t_k) = \vartheta_k / T_1$. Let us determine the sign of this functional ($t = t_k$) on optimal trajectories lying wholly within the phase constraints (Fig. 3; the trajectory $0, \delta^+, \delta^-, 0$). Suppose that the motion starts off with $u = +u_0$. The trajectory passing through the origin $y_1 = y_2 = 0$ ($F = 0$, $\delta = 0$) has, according to (3.1), the form

$$F + \frac{a_1}{a_2}\delta + \frac{aa_2 + a_1}{a_2^2}u_0 = \frac{aa_2 + a_1}{a_2^2}u_0 \exp\left(\frac{a_2}{u_0}\delta\right) \tag{5.1}$$

while the one passing through the point $F = 0$, $\delta = \delta^*$ has the form

$$F + \frac{a_1}{a_2}\delta - \frac{aa_2 + a_1}{a_2^2}u_0 = \left[\frac{a_1}{a_2}\delta^* - \frac{aa_2 + a_1}{a_2^2}u_0\right]\exp\left[-\frac{a_2}{u_0}(\delta - \delta^*)\right] \tag{5.2}$$

With the aid of (5.1), (5.2) we compute $J = J(0, \delta^+) + J(\delta^+, \delta^-) + J(\delta^-, 0)$, replacing $F(t)$ by $F(\delta)$,

$$J(\ldots) = \frac{1}{u_0}\left\{-\frac{a_1}{a_2}[(\delta^+)^2 - (\delta^-)^2]\right\} \tag{5.3}$$

To evaluate the sign of this quantity we substract (5.2) from (5.1), so as to obtain $\delta^+$, $\delta^-$, namely, the points of intersection of these trajectories. After some manipulations we obtain

$$\exp\frac{a_2}{u_0}\delta^{\pm} = 1 \mp \sqrt{1 + S(\delta^*)} \tag{5.4}$$

$$S(\delta^*) = \left[\frac{a_1a_2}{aa_2 + a_1}\frac{\delta_*}{u_0} - 1\right]\exp\left(\frac{a_2}{u_0}\delta^*\right)$$

Note that when $\delta^* < 0$ the curves do not intersect and that $\delta^+ > 0$ and $\delta^- < 0$.

For the roots to exist it is necessary that $1 + S(\delta^*) \leqslant 1$, $-1 \leqslant S(\delta^*) \leqslant 0$. Let us study the function $S(\delta^*)$. The derivative $dS / d\delta^*$ of this function preserves sign on the segment $0 \leqslant \delta^* \leqslant (aa_2 + a_1)u_0(a_1a_2)^{-1} - a_2u_0^{-1}$. From Fig. 3 it follows that the maximum value $\delta_m^* = (aa_2 + a_1)u_0(a_1a_2)^{-1}$. Hence, since on $0 \leqslant \delta^* \leqslant \delta_m^*$ the function $S(\delta^*)$ varies monotonically from $S(0) = -1$ to $S(\delta_m^*) = 0$, Eq. (5.4) has a real root and

$$\delta^+ - |\delta^-| = \delta^+ - \delta^- = \frac{u_0}{a_2} \ln [-S(\delta^*)] > 0$$

Thus, for $u^\circ(0) = + u_0$ we get that the plant with the optimal control found above reaches the terminal point $\vartheta_k > 0$. Hence, by virtue of symmetry, we obtain $u^\circ(0)\,\vartheta_k > 0$.

Note 3. In the case under consideration we obtain $|F(\delta^+)| - |F(\delta^-)| > 0$. This corresponds to the fact that if the first switching $(\delta^+)$ was within the phase constraints, then the second switching $(\delta^-)$ also will be within.

To prove the above-mentioned rule for choosing the initial control in the case when the optimal trajectory goes onto the phase constraints, we can either carry out analogous evaluations of the integral or be satisfied by the following arguments. Suppose that the motion starts off with $u^\circ(0) = + u_0$. Upon reaching the phase constraint $F = + F_0$ the system moves under this constraint with a monotonic growth of the coordinate $\delta(t)$ up to $\delta(\infty) = - a_2 F_0 a_1^{-1}$ (see (3.4)). If $- a_2 F_0 a_1^{-1} \leqslant \delta_0$, the trajectory can be situated on the constraint $F = + F_0$ for an arbitrarily long time, increasing in doing so the value of

$$\int_0^t b_2 F(t)\, dt$$

After coming off this constraint and moving on the terminal segment, even in the case of going onto the constraint $F = - F_0$, by virtue of the conditions

$$|\delta_1^-| < |\delta_1^+|, \qquad |\delta_2^+ - \delta_1^+| > |\delta_1^- - \delta_2^-|$$

$$|\delta_1^-| < \left|\frac{a_2}{a_1} F_0\right|$$

we find that the system remains with $F(t) \leqslant 0$ for a limited time. If, however, $-a_2 F_0 a_1^{-1} > \delta_0$, the system locates on the phase constraint $F = + F_0$ for a limited time and then crosses over the constraint $\delta(t) = \delta_0$. The coordinate $F(t)$ moving along this constraint decreases monotonically and tends to the value $F(\infty) = - a_1 \delta_0 a_2^{-1} > 0$. Thus, in this case too, the system can sustain the condition $F(t) > 0$ for an arbitrarily long time.

Thus, among the phase trajectories which have the structure established above for optimal trajectories, only the trajectories with $u(t)\,\vartheta_k|_{t=0} > 0$ ensure any value of

$$\int_0^{t_k} b_2 F(t)\, dt$$

of the sign required.

Note 4. In this paper we have not written out the explicit dependence of the optimal control $u^\circ(\ldots)$ on the plant's current phase coordinates. We doubt the advisability of determining this dependence in view of its unwieldiness. The expressions derived for the determination of the characteristic instants ($t_3$, $t_5$, etc.) in terms of the current phase coordinates and the expressions for the determination of the optimal control at the intermediate instants yield full information for the practical realization of the desired synthesis.

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